NoVaS Methods for Forecasting Financial Volatility

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Problem Statement

Basic Idea: Volatility forecasting plays an important role in financial econometrics. Previous works in this regime are mainly based on applying various GARCH-type models. However, it is hard for participants to choose a specific GARCH model that works for general cases and such traditional methods are unstable for dealing with high-volatile periods or using a small sample size. The recently proposed *Normalizing and Variance Stabilizing* (**NoVaS**) method is a more robust and accurate prediction technique based on the Model-free Prediction Principle of Politis (2015). This method is initially built by taking advantage of an inverse transformation which is based on the ARCH model. More importantly, this method does not rely on any assumption about the underlying distribution, so it is a *distribution-free* prediction technique.

Our contributions:

1. One drawback of the existing NoVaS-type methods is that the parameters of the transformation must obey a specific form, which decreases their flexibility. Inspired by the development of the ARCH model to the GARCH model, we attempt to build a novel NoVaS method derived by iterating the GARCH(1,1) structure.
2. The current NoVaS method does not consider the additional information in the prediction procedure. We derive the new NoVaS-type method with exogenous covariates to obtain point predictions and prediction intervals.

Model-free Prediction Principle

The Model-free Prediction Principle relies on four steps (Politis, 2015) :

1. Find an invertible transformation function h_T which transforms non-$i.i.d.$ samples $(Y_1,\ldots,Y_T)$ to $i.i.d.$ vector $(e_1,\ldots,e_T)\overset{i.i.d.}{\sim}F_e$ with possible explanatory variables $(X_1,\ldots,X_T)$.
2. Solve for Y_T in terms of $\boldsymbol{Y}_{T-1}:=(Y_1,\ldots,Y_{T-1})$, X_T and e_T, i.e., $Y_T=h_T^{-1}(\boldsymbol{Y}_{T-1},X_T,e_T)$.
3. Determine the future response $Y_f:=h_T^{-1}(\boldsymbol{Y}_T,X_f,e_f)$, where $e_f\sim F_e$ is independent with Y_f, X_f and $(e_1,\ldots,e_T)$.
4. Evaluate the whole distribution of Y_f by Monte Carlo (F_e is known) or Bootstrap (F_e is estimated).

NoVaS Transformation and Prediction

Transformation: The NoVaS transformation is a straightforward application of the Model-free Prediction Principle, which is based on the ARCH model as follows:

$$Y_t=W_t\sqrt{a+\sum_{i=1}^{p}a_iY_{t-i}^2}. \tag{1}$$

In Eq. (1), these parameters satisfy $a\geq 0$ and $a_i\geq 0$ for all $i=1,\ldots,p$ and $W_t\sim i.i.d.\ N(0,1)$. We express W_t in Eq. (1) using the following terms with one additional term:

$$W_t=\frac{Y_t}{\sqrt{\alpha s_{t-1}^2+\beta Y_t^2+\sum_{i=1}^{p}a_iY_{t-i}^2}}\ ;\ \text{for } t=p+1,\ldots,T; \tag{2}$$

where s_{t-1}^2 is the sample variance of $\{Y_i\}_{i=1}^{t-1}$. Subsequently, Eq. (2) can be considered a potential form of h_T in the *Model-free Prediction Principle*. All unknown coefficients are determined by:

1. Assuming $\alpha\neq 0,\beta=c',a_i=c'e^{-ci}$; for all $1\leq i\leq p, c'=\frac{1-\alpha}{\sum_{j=0}^{p}e^{-cj}}$.
2. Minimizing $|KURT(W_t)-3|$ w.r.t. to c for a grid values of α.

Prediction: We can write

$$Y_t=\sqrt{\frac{W_t^2}{1-\beta W_t^2}(\alpha s_{t-1}^2+\sum_{i=1}^{p}a_iY_{t-i}^2)}\ ;\ \text{for } t=p+1,\ldots,T. \tag{3}$$

We can easily obtain the analytical form of Y_{T+1}:

$$Y_{T+1}=\sqrt{\frac{W_{T+1}^2}{1-\beta W_{T+1}^2}(\alpha s_T^2+\sum_{i=1}^{p}a_iY_{T+1-i}^2)}. \tag{4}$$

Moreover, we can express Y_{T+h} as

$$Y_{T+h}=f(W_{T+1},\ldots,W_{T+h};\mathcal{F}_T)\ ;\ \text{for any } h\geq 1. \tag{5}$$

The analytical form of Y_{T+h} from the NoVaS transformation depends only on $i.i.d.$ $\{W_{T+1},\ldots,W_{T+h}\}$ and $\mathcal{F}_T$. The function f is known by expressing Y_{T+h} iteratively based on Eq. (4). So, we can generate M pseudo values $\{W_{T+1,m},\ldots,W_{T+h,m}\}_{m=1}^{M}$ by bootstrapping to approximate the distribution of Y_{T+h}. Moreover, the quantile prediction interval can also be built.

NoVaS Transformation Based on GARCH(1,1)

GARCH(1,1) model: $Y_t=\sigma_tW_t,\ \sigma_t^2=a+a_1Y_{t-1}^2+b_1\sigma_{t-1}^2$.

GARCH-NoVaS transformation:

$$W_t=\frac{Y_t}{\sqrt{\alpha s_{t-1}^2+\sum_{i=1}^{q}\tilde{c}_iY_{t-i}^2}}\ ;\ Y_t=\sqrt{W_t^2(\alpha s_{t-1}^2+\sum_{i=1}^{q}\tilde{c}_iY_{t-i}^2)}; \tag{6}$$

where $\{\tilde{c}_1,\ldots,\tilde{c}_q\}$ represents $\{a_1,a_1b_1\ ,a_1b_1^2,\ \ldots,a_1b_1^{q-1}\}$ scaled by multiplying a scalar $\frac{1-\alpha}{\sum_{j=1}^{q}a_1b_1^{j-1}}$ and the optimal combination of α,a_1,b_1 is selected by minimizing $|KURT(W_t)-3|$ w.r.t. a_1 and b_1 for a grid values of α.

GARCH-NoVaS prediction: Similarly with NoVaS prediction, we can express Y_{T+h}:

$$Y_{T+h}=f_{\text{GA}}(W_{T+1},\ldots,W_{T+h};\mathcal{F}_T)\ ;\ \text{for any } h\geq 1. \tag{7}$$

The distribution of Y_{T+h} can be approximated by the bootstrap technique again.

Comparison with NoVaS:

1. More parsimonious and more stable on prediction (see if $W_{T+1}^2\approx 1/\beta$).
2. More complete optimization search region compared to NoVaS minimization w.r.t. to c, i.e., more freedom to do optimization task.

NoVaS Transformation Based on GARCHX(1,1,m)

GARCHX model: $Y_t=\sigma_tW_t;\sigma_t^2=a+a_1Y_{t-1}^2+b_1\sigma_{t-1}^2+\boldsymbol{c}^T\boldsymbol{X}_{t-1}$. To guarantee the non-negativity of σ_t^2, we require $\boldsymbol{X}_{t-1}:=(X_1,\ldots,X_m)\geq 0$.

Existence of $h_T(\cdot)$ and $h_T^{-1}(\cdot)$: We assume

A1 The joint density of $\{Y_1,\cdots,Y_T\}$ exists.

A2 For exogenous random vector $\boldsymbol{X}:=\{X_1,\cdots,X_m\}$, the joint density $\{Y_1,\cdots,Y_T,X_1,\cdots,X_m\}$ exists for any $m\geq 1$.

Theorem: Under A1 and A2, there exists a function h_T such that $\boldsymbol{Z}=h_T((\boldsymbol{Y},\boldsymbol{X}))$ and the corresponding inverse function h_T^{-1} such that $(\tilde{\boldsymbol{Y}},\tilde{\boldsymbol{X}})=h_T^{-1}(\boldsymbol{Z});\boldsymbol{Z}\sim N(0,\boldsymbol{I}_{T+m})\,;\boldsymbol{Y}=(Y_1,\cdots,Y_T)$ and $\boldsymbol{X}=(X_1,\cdots,X_m)$ are any two random vectors; $(\tilde{\boldsymbol{Y}},\tilde{\boldsymbol{X}})$ have the same joint distribution of $(\boldsymbol{Y},\boldsymbol{X})$.

GARCHX(1,1,1)-NoVaS transformation:

$$W_t=\frac{Y_t}{\sqrt{\alpha s_{t-1,Y}+\beta s_{t-1,X}+\sum_{i=1}^{p}a_1b_1^{i-1}Y_{t-i}^2+\sum_{i=1}^{p}c_1b_1^{i-1}X_{t-i}}}.$$

GARCHX(1,1,1)-NoVaS prediction: Similar with NoVaS and GARCH-NoVaS predictions, future exogenous series $\boldsymbol{X}_{T+h}$ is assumed to be known or predicted separately.

Real Data Analysis

Data description: Grebe et al. (2024) assembled a data set of more than eight million German Twitter posts related to the war in Ukraine to construct a daily index of uncertainty. We utilize this (Ukraine) index to forecast national stock market volatility of Germany and its two neighbors Austria and Switzerland over the daily period of January 1, 2021, to February 28, 2023.

Moving-window prediction: We use $\{Y_1,\cdots,Y_T\}$ to predict $\{Y_{T+1}^2,\cdots,Y_{T+h}^2\}$, then we use $\{Y_2,\cdots,Y_{T+1}\}$ to predict $\{Y_{T+2}^2,\cdots,Y_{T+h+1}^2\}$, and so on until we reach the end of the sample.

Table 1. MSPE ratios of methods on aggregated moving window predictions with Ukraine index.

	Germany			Austria			Switzerland		
Aggregated Prediction steps	1	5	20	1	5	20	1	5	20
GA	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
GAX-Ukraine	1.011	1.023	0.989	1.002	1.003	0.994	1.003	1.008	1.005
GAX-NoVaS-Ukraine	1.038	1.141	0.898	1.017	0.984	0.822	1.010	1.100	0.879

See more data analyses from Wu and Karmakar (2021, 2023); Wu et al. (2025).

Discussion

1. The NoVaS transformation can be extended to incorporate other GARCH-type models for the volatility prediction tasks.
2. The NoVaS transformation can be extended to handle other financial data types, e.g., high-frequency data.
3. An efficient optimization algorithm is critical to the success of NoVaS.
4. The NoVaS idea could be applied to other tasks besides predictions.

References

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