MATH 170A Summer Session II 2024

Discussion 1

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Office hours: Th: 6:00 PM - 7:30 PM; F: 10:00 PM - 11:30 PM

Zoom ID: Access from Canvas

Discussion design:

- Talk about lecture materials (around 40 minutes)
  - Review key knowledge points
  - Go over some exercises
- Question session (around 10 minutes)

1 Matrix multiplication

1.1 Multiplying a Matrix by a Vector

Let's say we have a real matrix A and a vector x,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} ; \; x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

How can we compute Ax?

$$b = Ax = ?$$

Answer:

Example 1.1.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \\ 6 \end{bmatrix} = ?$$

1.2 Multiplying a Matrix by a Matrix

If A is an $n \times m$ matrix, and X is $m \times p$, i.e.,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} ; \; X = \begin{bmatrix} x_{11} & x_{12} & \cdots & a_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mp} \end{bmatrix}$$

we can get the product $B = AX$, which is $n \times p$. The (i, j) entry of B is

Remark 1.2 (Block Matrices and Block Matrix Operations)**.** *We can consider the block version of matrices and think about the operation rule with block matrices. For example, we can write:*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} ; \ X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

Then,

1.3 Implementation with Matlab

Actually, you do not need to install Mathlab on your PC. We can use the Matlab Web server through the link with your student account.

1.4 Matrix property

Theorem 1.3. *Let A be a square matrix. The following six conditions are equivalent:*

- *(a) A^{-1} exists.*
- *(b) There is no nonzero x such that $Ax =$ **0**.*
- *(c) The columns of A are linearly independent.*
- *(d) The rows of A are linearly independent.*
- *(e) $\det(A) \neq 0$.*
- *(f) Given any vector b, there is exactly one vector x such that $Ax = b$.*

Definition 1.4. *The vectors $v_1, \ldots, v_n$ are linearly independent if the only solution to $x_1 v_1 + \cdots + x_n v_n = 0$ (with $x_i \in \mathbb{R}$) is $x_1 = \cdots = x_n = 0$.*

Definition 1.5 (Invertible square matrix)**.** *An n-by-n square matrix A is **invertible** (also called nonsingular, nondegenerate), if there exists an n-by-n square matrix B such that*

$$AB = BA = I.$$

2 Linear and triangular systems

The Lower-Triangular Systems have the form as below:

$$\begin{aligned}
&g_{11}y_1 \quad = b_1\\
&g_{21}y_1 + g_{22}y_2 = b_2\\
&g_{31}y_1 + g_{32}y_2 + g_{33}y_3 = b_3\\
&g_{n1}y_1 + g_{n2}y_2 + g_{n3}y_3 + \cdots + g_{nn}y_n = b_n.
\end{aligned}$$

We have to solve:

$$Gy = b;$$

where G is a lower triangular matrix:

$$G = \begin{bmatrix} g_{11} & 0 & 0 & \cdots & 0 \\ g_{21} & g_{22} & 0 & \cdots & 0 \\ g_{31} & g_{32} & g_{33} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ g_{n1} & g_{n2} & g_{n3} & \cdots & g_{nn} \end{bmatrix}$$

We can solve for y by forward substitution (row- or column-oriented version)
Similarly, the Upper-Triangular Systems have the form as below:

$$\begin{aligned}
&u_{11}x_1 + u_{12}x_2 + \cdots + u_{1,n-1}x_{n-1} + u_{1,n}x_n = y_1\\
&u_{22}x_2 + \cdots + u_{2,n-1}x_{n-1} + u_{2n}x_n = y_2\\
&u_{n-1,n-1}x_{n-1} + u_{n-1,n}x_n = y_{n-1}\\
&u_{nn}x_n = y_n.
\end{aligned}$$

We can solve $Uy = b$ by back substitution (row- or column-oriented version).

Example 2.1.

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 5 \\ 3 \end{bmatrix}$$

Solve it by column-oriented back substitution method: