MATH 170A Summer Session II 2024

Discussion 10

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1 Definition reviews

1.1 Eigenvalues and eigenvectors

Let $A \in \mathbb{C}^{n\times n}$. A vector $v \in \mathbb{C}^n$ is called an eigenvector of A if $v \neq 0$ and Av is a multiple of v; that is, there exists a $\lambda \in \mathbb{C}$ such that

$$Av = \lambda v$$

The scalar λ is called the **eigenvalue** of A associated with the **eigenvector** v. On the other hand, v is called an eigenvector of A associated with the eigenvalue λ. The pair (λ, v) is called an eigenpair of A. Each eigenvalue is associated with many eigenvectors.

Theorem 1.1. *λ is an eigenvalue of A if and only if*

$$\det(A - \lambda I) = 0$$

Moreover, $p(\lambda) := \det(A - \lambda I)$ which a polynomial in λ of degree n. It is called the characteristic polynomial of A. Eigenvalues are roots of $p(\lambda)$.

Exercise 1.2. *Let $A \in \mathbb{C}^{n\times n}$, use the characteristic equation to show that A and A^T have the same eigenvalues.*

Exercise 1.3. *Let*

$$B = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$$

Show that the characteristic polynomial of B is $(\lambda - 1)(\lambda - 2)$ and then find eigenvalues.

Theorem 1.4. *Let $T \in \mathbb{C}^{n\times n}$ be a (lower-or upper-) triangular matrix. Then the eigenvalues of T are the main-diagonal entries $t_{11}, \ldots, t_{nn}$.*

Theorem 1.5. *Let $v_1, \ldots, v_k$ be eigenvectors of A associated with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. Then $v_1, \ldots, v_k$ are linearly independent.*

Definition 1.6. *let $A \in \mathbb{C}^{n\times n}$.*

1 *A is semisimple if A has n linearly independent eigenvectors.*

2 *A is defective if A is not semisimple.*

3 *A is simple if A has no repeated eigenvalue.*

1.2 Power method

It follows that there is no general formula for the eigenvalues of an $n \times n$ matrix if $n > 4$ since it is hard to solve the root of a characteristic polynomial when the degree is large. Thus, we need to use some numerical method.

Let $A \in \mathbb{C}^{n\times n}$. We assume that A is semisimple. This means that A has a set of n linearly independent eigenvectors $v_1, \ldots, v_n$, which must then form a basis for $\mathbb{C}^n$. Let $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues associated with $v_1, \ldots, v_n$, respectively. Let us assume that the vectors are ordered so that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$. If $|\lambda_1| > |\lambda_2|$, λ_1 is called the **dominant eigenvalue** and v_1 is called a **dominant eigenvector** of A.

If A has a dominant eigenvalue, then we can find it and an associated dominant eigenvector by the power method. The basic idea is to pick a (random) vector q which can be expressed as

$$q = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

Then, we can form the sequence

$$q, Aq, A^2q, A^3q, \dots$$

Consequently, we can show $A^j q/\lambda_1^j$ converges to $c_1 v_1$. In other words, $A^j q$ indicates the direction of the eigenvector v_1 which is the thing we are interested in. Since λ_1 is unknown in practice, we can divide a convenient scaling factor for q_j before each iterative step.

Example 1.7 (Textbook Example 5.3.5)**.** *Use the power method to calculate a dominant eigenvector of*

$$A = \begin{bmatrix} 9 & 1 \\ 1 & 2 \end{bmatrix}$$

If we start with the vector $q_0 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$*, then on the first step we have* $Aq_0 = \begin{bmatrix} 10 & 3 \end{bmatrix}^T$*. Dividing by the scale factor* $\sigma_1 = 10$*, we get* $q_1 = \begin{bmatrix} 1 & 0.3 \end{bmatrix}^T$*. Then* $Aq_1 = \begin{bmatrix} 9.3 & 1.6 \end{bmatrix}^T, \sigma_2 = 9.3$*, and* $q_2 = \begin{bmatrix} 1 & 0.172034 \end{bmatrix}^T$*. Subsequent iterates are listed in Table 5.1. Only the second component of each* q_j *is listed because the first component is always 1. We see that after 10 iterations the sequence of vectors* (q_j) *has converged to six decimal places. Thus (to six decimal places)*

$$v_1 = \begin{bmatrix} 1.0 \\ 0.140055 \end{bmatrix}$$

The sequence σ_j *converges to the dominant eigenvalue* $\lambda_1 = 9.140055$*.*

j	σ_j	q_j (second component)
3	9.172043	0.146541
4	9.146541	0.141374
5	9.141374	0.140323
6	9.140323	0.140110
7	9.140110	0.140066
8	9.140066	0.140057
9	9.140057	0.140055
10	9.140055	0.140055