MATH 170A Summer Session II 2024

Discussion 2

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1 Definition reviews

Definition 1.1 (Positive definite matrix)**.** *If A is $n \times n$ real and symmetric matrix satisfying*

$$x^T Ax > 0$$

for all nonzero $x \in \mathbb{R}^n$, then A is positive definite.

A related theorem is:

Theorem 1.2. *Let M be any $n \times n$ nonsingular matrix, and let $A = M^T M$. Then A is positive definite.*

Theorem 1.3 (Cholesky Decomposition Theorem)**.** *Let A be a positive definite matrix. Then we can decompose A uniquely into a product*

$$A = R^T R \quad \textit{(Cholesky Decomposition)}$$

such that R is upper triangular and has all main diagonal entries r_{ii} positive. R is called the Cholesky factor of A.

A related proposition is:

Proposition 1.4. *If matrix A has a Cholesky factor, then A is positive definite. If A is positive definite, then A must have a Cholesky factor*

2 Matrix transpose

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(cA)^T = cA^T$
- $(AB)^T = B^T A^T$
- $(A^T)^{-1} = (A^{-1})^T$, if A^{-1} exists.

3 Exercises

Exercise 3.1. *Find the range of t such that A is positive definite*

$$A = \begin{bmatrix} 16 & 4 & 8 & 4 \\ 4 & 10 & 8 & 4 \\ 8 & 8 & 12 & 10 \\ 4 & 4 & 10 & t \end{bmatrix}$$

Exercise 3.2. *Let* $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ *be a* 2×2 *block matrix, for submatrices* $A_{11} \in \mathbb{R}^{n_1\times n_1}$, $A_{12} \in \mathbb{R}^{n_1\times n_2}$, $A_{21} \in \mathbb{R}^{n_2\times n_1}$, $A_{22} \in \mathbb{R}^{n_2\times n_2}$. *If* A *is positive definite, show that* A_{22} *is invertible and the matrix* $B := A_{11} - A_{12}A_{22}^{-1}A_{21}$ *is also positive definite.*