MATH 170A Summer Session II 2024

Discussion 3

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1 Definition reviews

1.1 LU Decomposition

Proposition 1.1. *If matrix A has a Cholesky factor, then A is positive definite. If A is positive definite, then A must have a Cholesky factor. Moreover, the Cholesky Decomposition is unique.*

Definition 1.2. *(Gaussian elimination) The Gaussian elimination is used to simplify the computation of solving linear system $Ax = b$ without assuming symmetry or positive definiteness*

Remark 1.3. *A positive-definite matrix is always invertible. However, an invertible matrix may not be positive definite; see the matrix below:*

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Definition 1.4. *(Gaussian elimination without row interchanges) The Gaussian elimination can be performed with **only** operation "Add a multiple of one row to another row" if*

$$A_k \text{ is nonsingular} \quad k = 1, \ldots, n-1$$

A_k is the k-th leading principal submatrix of A.

Theorem 1.5. *(LU Decomposition) Let A be an $n \times n$ matrix whose leading principal submatrices are all nonsingular. Then A can be decomposed in a unique way into a product*

$$A = LU,$$

such that L is unit lower triangular and U is upper triangular.

Theorem 1.6. *(LDV Decomposition Theorem) Let A be an $n \times n$ matrix whose leading principal submatrices are all nonsingular. Then A can be decomposed in exactly one way as a product*

$$A = LDV,$$

such that L is unit lower triangular, D is diagonal, and V is unit upper triangular.

Proof. We need first to show such LDV decomposition exists. Then, we need to show it is unique. $\square$

Theorem 1.7. *(LDL^T Decomposition Theorem) Let A be a* ***symmetric*** *matrix whose leading principal submatrices are nonsingular. Then A can be expressed in exactly one way as a product*

$$A = LDL^T,$$

such that L is unit lower triangular and D is diagonal.

Remark 1.8. *Any square matrix A admits LUP and PLU factorizations.*

Exercise 1.9. *Let A be a nonsingulor matrix. $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, A_{11} is nonsingular*

- *(a) Show that*
$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ M & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}$$

- *(b) Both A_{11} and $\tilde{A}_{22}$ have LU decompositions $A_{11} = L_1U_1$ and $\tilde{A}_{22} = L_2U_2$. Show that*
$$A = \begin{bmatrix} L_1 & 0 \\ ML_1 & L_2 \end{bmatrix} \begin{bmatrix} U_1 & L_1^{-1}A_{12} \\ 0 & U_2 \end{bmatrix}$$

  This is the LU decomposition of A.

1.2 Matrix and vector norms

Definition 1.10 (Vector norm)**.** *A norm on $\mathbb{R}^n$ is a function that assigns to each $x \in \mathbb{R}^n$ a non-negative real number $\|x\|$, called the norm of x, such that the following four properties are satisfied for all $x, y \in \mathbb{R}^n$ and all $\alpha \in \mathbb{R}$:*

$$
\begin{aligned}
&\|x\| \geq 0, \forall x \in \mathbb{R}^n \\
&\|x\| = 0 \textit{ implies } x = 0 \\
&\|\alpha x\| = |\alpha| \|x\| \\
&\|x + y\| \leq \|x\| + \|y\| \textit{ (triangle inequality)}
\end{aligned}
$$

Example 1.11. *Some common norms are below:*

- *1-norm:* $\|x\|_1 = \sum_{i=1}^n |x_i|$.
- *2-norm:* $\|x\|_2 = \left(\sum_{i=1}^n |x_i|^2\right)^{1/2}$.
- *p-norm:* $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$, $p \geq 1$.
- ∞*-norm:* $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$

Theorem 1.12 (Cauchy-Schwarz inequality)**.** *For all $x, y \in \mathbb{R}^n$, we have*

$$\left| \sum_{i=1}^n x_i y_i \right| \leq \left(\sum_{i=1}^n x_i^2 \right)^{1/2} \left(\sum_{i=1}^n y_i^2 \right)^{1/2}$$

Definition 1.13 (Matrix norm)**.** *A matrix norm is a function that assigns to each $A \in \mathbb{R}^{n \times n}$ a real number $\|A\|$, called the norm of A. Specifically, for all $A, B \in \mathbb{R}^{n \times n}$ and all $\alpha \in \mathbb{R}$,*

$$
\begin{aligned}
&\|A\| \geq 0 \\
&\|A\| = 0 \textit{ implies } A = 0 \\
&\|\alpha A\| = |\alpha| \|A\| \\
&\|A + B\| \leq \|A\| + \|B\| \\
&\|AB\| \leq \|A\| \|B\| \quad \textit{(operator norm)}
\end{aligned}
$$

Definition 1.14 (Induced matrix norm)**.** *The matrix norm induced by a vector norm $\|\cdot\|_v$ is defined by*

$$\|A\|_M = \max_{x \neq 0} \frac{\|Ax\|_v}{\|x\|_v}$$

Exercise 1.15. *Prove that the induced matrix norm satisfies:* $\|A\| = \max_{\|x\|=1} \|Ax\|$

Definition 1.16 (matrix p-norm)**.**

$$\|A\|_p = \max_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$$

Theorem 1.17. *Matrix 1-norm and* ∞*-norm satisfy below equations*

(a) $\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}|$ *(column-sum norm).*

(b) $\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|$ *(row-sum norm).*

Proof. (a) is proved in class. The proof of (b) is below:

□