MATH 170A Summer Session II 2024

Discussion 4

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1 Definition reviews

1.1 Matrix and vector norms

Definition 1.1 (Vector norm). *A norm on $\mathbb{R}^n$ is a function that assigns to each $x \in \mathbb{R}^n$ a non-negative real number $\|x\|$, called the norm of x, such that the following four properties are satisfied for all $x, y \in \mathbb{R}^n$ and all $\alpha \in \mathbb{R}$:*

$$\begin{aligned}
&\|x\| \geq 0, \forall x \in \mathbb{R}^n \\
&\|x\| = 0 \text{ implies } x = 0 \\
&\|\alpha x\| = |\alpha| \|x\| \\
&\|x + y\| \leq \|x\| + \|y\| \text{ (triangle inequality)}
\end{aligned}$$

Theorem 1.2 (Cauchy-Schwarz inequality). *For all $x, y \in \mathbb{R}^n$, we have*

$$\left| \sum_{i=1}^n x_i y_i \right| \leq \left(\sum_{i=1}^n x_i^2 \right)^{1/2} \left(\sum_{i=1}^n y_i^2 \right)^{1/2}$$

Definition 1.3 (Matrix norm). *A matrix norm is a function that assigns to each $A \in \mathbb{R}^{n\times n}$ a real number $\|A\|$, called the norm of A. Specifically, for all $A, B \in \mathbb{R}^{n\times n}$ and all $\alpha \in \mathbb{R}$,*

$$\begin{aligned}
&\|A\| \geq 0 \\
&\|A\| = 0 \text{ implies } A = 0 \\
&\|\alpha A\| = |\alpha| \|A\| \\
&\|A + B\| \leq \|A\| + \|B\| \\
&\|AB\| \leq \|A\| \|B\| \quad \text{(operator norm)}
\end{aligned}$$

Definition 1.4 (Induced matrix norm). *The matrix norm induced by a vector norm $\|\cdot\|_v$ is defined by*

$$\|A\|_M = \max_{x\neq 0} \frac{\|Ax\|_v}{\|x\|_v}$$

Exercise 1.5. *Show that* $\sqrt{tr(A^T A)}$ *defines a matrix norm function.*

Exercise 1.6. *Show that* $\|x\|_1 \leq \sqrt{n}\|x\|_2$.

1.2 Condition number

Definition 1.7 (Condition number (informal))**.** *Condition number is a useful measure of the sensitivity of the linear system $Ax = b$. For example, if there is a perturbation in the b vector, $i.e.$, we actually have $b + \delta b$. δb could be some rounding error when the vector b is stored in computer.*

We are interested in exploring if $\|\delta x\|/\|x\|$ is also small when $\|\delta b\|/\|b\|$ is small.

The condition number of a matrix A is:

$$\kappa(A) = \|A\| \left\|A^{-1}\right\| \geq 1.$$

Theorem 1.8. *Let A be nonsingular, and let x and $\hat{x} = x + \delta x$ be the solutions of $Ax = b$ and $A\hat{x} = b + \delta b$, respectively. Then*

$$\frac{\|\delta x\|}{\|x\|} \leq \kappa(A) \frac{\|\delta b\|}{\|b\|}$$

Definition 1.9 (maximum and minimum magnification)**.**

$$\operatorname{maxmag}(A) = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \|A\|$$

and

$$\operatorname{minmag}(A) = \min_{x \neq 0} \frac{\|Ax\|}{\|x\|}.$$

Thus, we have:

$$\kappa(A) = \frac{\operatorname{maxmag}(A)}{\operatorname{minmag}(A)}$$

Proposition 1.10.

$$\operatorname{maxmag}(A) = \frac{1}{\operatorname{minmag}\left(A^{-1}\right)} \quad \textit{and} \quad \operatorname{maxmag}\left(A^{-1}\right) = \frac{1}{\operatorname{minmag}(A)}$$

From textbook:

An ill-conditioned matrix is one for which the maximum magnification is much larger than the minimum magnification.

If the matrix A is nonzero but singular, then there exists $x \neq 0$ such that $Ax = 0$. Thus $\operatorname{minmag}(A) = 0$, and it is reasonable to say that $\kappa(A) = \infty$. That is, we view singularity as the extreme case of ill-conditioning. Reversing the viewpoint, we can say that an ill-conditioned matrix is one that is "nearly" singular.

Exercise 1.11. *Using* 1*-norm, find the condition number of the following matrix, where* $0 < \varepsilon < 1$*:*

$$A = \begin{bmatrix} 1 & 0 \\ 1 & \varepsilon \end{bmatrix}$$

Explain what happens to the condition number $\kappa(A)$*, when* ε *varies*

Exercise 1.12. *Let* $A = \begin{bmatrix} 1001 & 1000 \\ 1000 & 1001 \end{bmatrix}$. *Find nonzero vectors* $b, \delta b, x, \delta x$ *such that* $Ax = b$ $A(x + \delta x) = b + \delta b$ *and the product* $\frac{\|\delta x\|_\infty}{\|x\|_\infty} \cdot \frac{\|b\|_\infty}{\|\delta b\|_\infty}$ *is maximum.*