MATH 170A Summer Session II 2024

Discussion 5

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1 Definition reviews

1.1 Perturbing the coefficient matrix

Up to this point we have considered only the effect of perturbing b in the system $Ax = b$. We may also consider perturbations of A i.e., the comparison of two systems $Ax = b$ and $(A + \delta A)\hat{x} = b$, where $\|\delta A\|/\|A\|$ is small.

Theorem 1.1. *If A is nonsingular and*

$$\frac{\|\delta A\|}{\|A\|} < \frac{1}{\kappa(A)}$$

then $A + \delta A$ is nonsingular.

Theorem 1.2. *Let A be nonsingular, let $b \neq 0$, and let x and $\hat{x} = x + \delta x$ be solutions of $Ax = b$ and $(A + \delta A)\hat{x} = b$, respectively. Then,*

$$\frac{\|\delta x\|}{\|\hat{x}\|} \leq \kappa(A)\frac{\|\delta A\|}{\|A\|}$$

Remark 1.3. *The above theorem is about the relationship between δx and $\hat{x}$ and no nonsingularity assumption of $A + \delta A$ is made.*

Theorem 1.4. *If A is nonsingular, $\|\delta A\|/\|A\| < 1/\kappa(A), b \neq 0, Ax = b$, and $(A+\delta A)(x+\delta x) = b$, then*

$$\frac{\|\delta x\|}{\|x\|} \leq \frac{\kappa(A)\frac{\|\delta A\|}{\|A\|}}{1 - \kappa(A)\frac{\|\delta A\|}{\|A\|}}$$

1.2 Inner product

Given two vectors x and y in $\mathbb{R}^n$, we can define inner product of x and y, which satisfies below properties:

$$\begin{aligned}
\langle x, y\rangle &= \langle y, x\rangle \\
\langle \alpha_1 x_1 + \alpha_2 x_2, y\rangle &= \alpha_1 \langle x_1, y\rangle + \alpha_2 \langle x_2, y\rangle \\
\langle x, \alpha_1 y_1 + \alpha_2 y_2\rangle &= \alpha_1 \langle x, y_1\rangle + \alpha_2 \langle x, y_2\rangle \\
\langle x, x\rangle &\geq 0 \quad \text{with equality if and only if } x = 0.
\end{aligned}$$

The Euclidean norm (2-norm) can be induced from inner product in the space $\mathbb{R}^n$:

$$\|x\|_2 = \sqrt{\langle x, x\rangle}$$

The Cauchy-Schwarz inequality can be stated more concisely in terms of the inner product and the Euclidean norm:

$$|\langle x, y\rangle| \leq \|x\|_2 \|y\|_2$$

We can define the angle between (nonzero) x and $y \in \mathbb{R}^n$ to be

$$\theta = \arccos \frac{\langle x, y\rangle}{\|x\|_2 \|y\|_2} \ ; \ \cos\theta = \frac{\langle x, y\rangle}{\|x\|_2 \|y\|_2}$$

If $n = 2$, we are familiar with such a result due to the geometry meaning.

1.3 Orthogonal matrix

Definition 1.5. *A matrix $Q \in \mathbb{R}^{n \times n}$ is said to be orthogonal if $QQ^T = I$. This equation says that Q has an inverse, and $Q^{-1} = Q^T$. Since a matrix always commutes with its inverse, we have $Q^T Q = I$ as well. For square matrices, the below equations are equivalent to defining the orthogonal matrix.*

$$QQ^T = I \quad Q^T Q = I \quad Q^T = Q^{-1}$$

Theorem 1.6. *If $Q \in \mathbb{R}^{n \times n}$ is orthogonal, then for all $x, y \in \mathbb{R}^n$,*

- $\langle Qx, Qy\rangle = \langle x, y\rangle$,
- $\|Qx\|_2 = \|x\|_2$.

Exercise 1.7. *What conditions must a and b satisfy for the matrix $A = \begin{bmatrix} a+b & b-a \\ a-b & a+b \end{bmatrix}$ to be orthogonal?*

1.4 Rotators

Rotate a vector x by angle θ counterclockwise:

$$y = Qx \; ; \; Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Rotate a vector x by angle θ clockwise:

$$y = Qx \; ; \; Q = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

1.5 Reflectors

Theorem 1.8. *Let $u \in \mathbb{R}^n$ with $\|u\|_2 = 1$, and define $Q \in \mathbb{R}^{n\times n}$ by $Q = I - 2uu^T$. Then*

- *(a) $Qu = -u$.*
- *(b) $Qv = v$ if $\langle u, v\rangle = 0$.*
- *(c) $Q = Q^T$ (Q is symmetric).*
- *(d) $Q^T = Q^{-1}$ (Q is orthogonal).*
- *(e) $Q^{-1} = Q$.*

Theorem 1.9. *Let $x, y \in \mathbb{R}^n$ with $x \neq y$ but $\|x\|_2 = \|y\|_2$. Then there is a unique reflector Q such that $Qx = y$.*

1.6 QR decomposition

Theorem 1.10. *Let $A \in \mathbb{R}^{n\times n}$. Then there exists an orthogonal matrix Q and an upper triangular matrix R such that $A = QR$.*

Theorem 1.11. *Let $A \in \mathbb{R}^{n\times n}$ be nonsingular. There exist unique $Q, R \in \mathbb{R}^{n\times n}$ such that Q is orthogonal, R is upper triangular with positive main-diagonal entries, and $A = QR$.*

Exercise 1.12. *Do QR decomposition by Reflectots for the matrix*

$$A = \begin{bmatrix} 12 & -51 & 4 \\ 6 & 167 & -68 \\ -4 & 24 & -41 \end{bmatrix}$$