MATH 170A Summer Session II 2024

Discussion 6

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1 Definition reviews

1.1 Gram-Schmidt Process

Definition 1.1 (Orthonormal vectors)**.** *A set of vectors $q_1, q_2, \dots, q_k \in \mathbb{R}^n$ is said to be orthonormal if the vectors are pairwise orthogonal, and each vector has Euclidean norm 1; that is,*

$$\langle q_i, q_j \rangle = \begin{cases} 0 & \textit{if } i \neq j \\ 1 & \textit{if } i = j \end{cases}$$

Theorem 1.2. *Let $Q \in \mathbb{R}^{n\times n}$. Then Q is an orthogonal matrix if and only if its columns (rows) form an orthonormal set.*

Definition 1.3 (Isometric matrix)**.** *The matrix $Q \in \mathbb{R}^{m\times n}$, $m \geq n$ will be called isometric if its columns are orthonormal, i.e., Q is an isometric if and only if $Q^T Q = I$.*

Theorem 1.4 (Condensed QR Decomposition)**.** *Let $A \in \mathbb{R}^{m\times n}, m \geq n$. Then there exist matrices $\hat{Q}$ and $\hat{R}$ such that $\hat{Q} \in \mathbb{R}^{m\times n}$ is an isometric matrix, $\hat{R} \in \mathbb{R}^{n\times n}$ is upper triangular, and*

$$A = \hat{Q}\hat{R}$$

Definition 1.5 (The procedure of Modified Gram-Schmidt Process)**.**

Exercise 1.6. *Find the QR decomposition of A*

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = QR$$

1.2 Solution of least square problem

Consider an overdetermined system

$$Ax = b, \quad A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, m > n$$

Usually, $Ax = b$ does not have an exact solution. Thus, we try to find a solution $\hat{x}$ that minimizes $\|A\hat{x} - b\|_2$.

Theorem 1.7. *Let $A \in \mathbb{R}^{m\times n}, m > n$. Then there exist $Q \in \mathbb{R}^{m\times m}$ and $R \in \mathbb{R}^{m\times n}$, such that Q is orthogonal and $R = \begin{bmatrix} \hat{R} \\ 0 \end{bmatrix}$, where $\hat{R} \in \mathbb{R}^{n\times n}$ is upper triangular, and $A = QR$.*

Theorem 1.8. *Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m, m > n$, and suppose A has full rank. Then the least squares problem for the overdetermined system $Ax = b$ has a unique solution, which can be found by solving the nonsingular system $\hat{R}x = \hat{b}_1$, where $\begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \end{bmatrix} = Q^T b, \hat{R} \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{m\times m}$ are as in Theorem 1.6.*

Remark 1.9. *If $\operatorname{rank}(A) < n. \exists P \in \mathbb{R}^{n\times n}$ such that*

$$A \cdot P = Q \begin{bmatrix} R_{11} & R_{12} \\ 0 & 0 \end{bmatrix}$$

$$\hat{x} = P \cdot \hat{y} = P \cdot \begin{bmatrix} R_{11}^{-1} \cdot \hat{b}_1 \\ 0 \end{bmatrix}$$

Exercise 1.10. *Find the least square solution to the overdetermined system*

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$$