MATH 170A Summer Session II 2024

Discussion 7

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1 Definition reviews

1.1 QR decomposition

For a matrix $A \in \mathbb{R}^{m\times n}$; $m \geq n$.

- If Rank(A) = n, then $A = QR$; $Q \in \mathbb{R}^{m\times m}$ and $R \in \mathbb{R}^{m\times n}$, such that Q is orthogonal and $R = \begin{bmatrix} \hat{R} \\ 0 \end{bmatrix}$, where $\hat{R} \in \mathbb{R}^{n\times n}$ is upper triangular.
- If Rank(A) = $r < n$,
$$A \cdot P = Q \begin{bmatrix} R_{11} & R_{12} \\ 0 & 0 \end{bmatrix};$$
$P \in \mathbb{R}^{n\times n}$; $R_{11} \in \mathbb{R}^{r\times r}$; $Q \in \mathbb{R}^{m\times m}$ and $R \in \mathbb{R}^{m\times n}$. So, we need the column pivoting to achieve QR decomposition.

1.2 Singular Value Decomposition (SVD)

SVD is more powerful than QR decomposition.

Theorem 1.1 (SVD Theorem)**.** *Let $A \in \mathbb{R}^{m\times n}$ be a nonzero matrix with rank r. Then A can be expressed as a product*

$$A = U\Sigma V^T,$$

where $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ are orthogonal, and $\Sigma \in \mathbb{R}^{m\times n}$ is a nonsquare "diagonal" matrix

$$\Sigma = \begin{bmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_r & & \\ & & & & 0 & \\ & & & & & \ddots \end{bmatrix} \qquad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$$

Definition 1.2. *Some terms in SVD*

- $\sigma_1, \ldots, \sigma_r$ *of* Σ *are uniquely determined, and they are called the singular values of* A.
- *The columns of* U *are orthonormal vectors called right singular vectors of* A
- *The columns of* V *are called left singular vectors of* A

Theorem 1.3 (Geometric SVD Theorem)**.** *Let* $A \in \mathbb{R}^{m \times n}$ *be a nonzero matrix with rank* r. *Then* $\mathbb{R}^n$ *has an orthonormal basis* $v_1, \ldots, v_n, \mathbb{R}^m$ *has an orthonormal basis* $u_1, \ldots, u_m$*, and there exist* $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$ *such that*

$$Av_i = \begin{cases} \sigma_i u_i & i = 1, \ldots, r \\ 0 & i = r+1, \ldots, n \end{cases} \qquad A^T u_i = \begin{cases} \sigma_i v_i & i = 1, \ldots, r \\ 0 & i = r+1, \ldots, m \end{cases}$$

Theorem 1.4. *The SVD displays orthonormal bases for the four fundamental subspaces* $\mathcal{R}(A)$, $\mathcal{N}(A), \mathcal{R}\left(A^T\right)$*, and* $\mathcal{N}\left(A^T\right)$*:*

$$\begin{aligned} \mathcal{R}(A) &= \operatorname{span}\{u_1, \ldots, u_r\} \\ \mathcal{N}(A) &= \operatorname{span}\{v_{r+1}, \ldots, v_n\} \\ \mathcal{R}\left(A^T\right) &= \operatorname{span}\{v_1, \ldots, v_r\} \\ \mathcal{N}\left(A^T\right) &= \operatorname{span}\{u_{r+1}, \ldots, u_m\} \end{aligned}$$

where $\mathcal{R}(A) = \{y | y = Ax, x \in \mathbb{R}^n\}$*, i.e., range space of A;* $\mathcal{N}(A) = \{x \in \mathbb{R}^n | Ax = 0\}$*, i.e., the null space of A.*
In addition, Rank(A) = Dim($\mathcal{R}(A)$*); Rank(A) = rank(*Σ*) =* r*, i.e., the number of non-zero singular values.*

Similar to the condensed version of QR decomposition, we also have the condensed version of SVD:

Theorem 1.5 (Condensed SVD Theorem)**.** *Let* $A \in \mathbb{R}^{m \times n}$ *be a nonzero matrix of rank* r. *Then there exist* $\hat{U} \in \mathbb{R}^{m \times r}, \hat{\Sigma} \in \mathbb{R}^{r \times r}$*, and* $\hat{V} \in \mathbb{R}^{n \times r}$ *such that* $\hat{U}$ *and* $\hat{V}$ *are isometries,* $\hat{\Sigma}$ *is a diagonal matrix with main-diagonal entries* $\sigma_1 \geq \ldots \geq \sigma_r > 0$*, and*

$$A = \hat{U}\hat{\Sigma}\hat{V}^T$$

Theorem 1.6. *Let* $A \in \mathbb{R}^{m \times n}$ *have singular values* $\sigma_1 \geq \sigma_2 \geq \ldots \geq 0$. *Then* $\|A\|_2 = \sigma_1$.

Exercise 1.7. *Let* $A = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix}$. *Find the singular values of* A.

Exercise 1.8. *Let* $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$ *(a) Find the rank of* A*; (b) Find a condensed SVD for* A.

Exercise 1.9. *Show that if* Q *is orthogonal, then* $\|Q\|_2 = 1$.

Exercise 1.10. *Lee* $A = R^\top R$ *be its Cholesky decomposition, where* $A \in \mathbb{R}^{n\times n}$ *is a positive definite matrix. Show that* $\kappa_2(A) = (\kappa_2(R))^2$.