MATH 170A Summer Session II 2024

Discussion 8

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1 Definition reviews

1.1 Singular Value Decomposition (SVD)

SVD is more powerful than QR decomposition.

Theorem 1.1 (SVD Theorem)**.** *Let $A \in \mathbb{R}^{m\times n}$ be a nonzero matrix with rank r. Then A can be expressed as a product*

$$A = U\Sigma V^T,$$

where $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ are orthogonal, and $\Sigma \in \mathbb{R}^{m\times n}$ is a nonsquare "diagonal" matrix

$$\Sigma = \begin{bmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_r & & \\ & & & & 0 & \\ & & & & & \ddots \end{bmatrix} \quad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$$

Theorem 1.2 (Condensed SVD Theorem)**.** *Let $A \in \mathbb{R}^{m\times n}$ be a nonzero matrix of rank r. Then there exist $\hat{U} \in \mathbb{R}^{m\times r}, \hat{\Sigma} \in \mathbb{R}^{r\times r}$, and $\hat{V} \in \mathbb{R}^{n\times r}$ such that $\hat{U}$ and $\hat{V}$ are isometries, $\hat{\Sigma}$ is a diagonal matrix with main-diagonal entries $\sigma_1 \geq \ldots \geq \sigma_r > 0$, and*

$$A = \hat{U}\hat{\Sigma}\hat{V}^T$$

Theorem 1.3 (Best low rank approximation)**.** *Let $A \in \mathbb{R}^{m\times n}$ with $\operatorname{rank}(A) = r > 0$. Let $A = U\Sigma V^T$ be the SVD of A, with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$. For $k = 1, \ldots, r-1$, define $A_k = U\Sigma_k V^T$, where $\Sigma_k \in \mathbb{R}^{m\times n}$ is the diagonal matrix diag $\{\sigma_1, \ldots, \sigma_k, 0 \ldots, 0\}$. Then $\operatorname{rank}(A_k) = k$, and*

$$\sigma_{k+1} = \|A - A_k\|_2 = \min\{\|A - B\|_2 \mid \operatorname{rank}(B) \leq k\}$$

That is, of all matrices of rank k or less, A_k is closest to A.

Exercise 1.4. *$A = U\Sigma V^\top$. Determine the SVD of matrix $\left(A^\top A\right)^{-1} A^\top$ and $A\left(A^\top A\right)^{-1}$*

Exercise 1.5.

$$A = \begin{bmatrix} 0 & 1 & 4 & 2/3 \\ -1/3 & 0 & -4 & 2/3 \\ 2/3 & 2 & -2 & 0 \\ 2/3 & -2 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 9 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -2/3 & 2/3 & -1/3 \\ 2 & 1 & -2 \\ -1/3 & -2/3 & -2/3 \end{bmatrix}$$

Find the best rank-one approximation to A.