MATH 170A Summer Session II 2024

Discussion 9

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1 Definition reviews

1.1 Singular Value Decomposition (SVD)

Theorem 1.1 (SVD Theorem). *Let $A \in \mathbb{R}^{m\times n}$ be a nonzero matrix with rank r. Then A can be expressed as a product*

$$A = U\Sigma V^T,$$

where $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ are orthogonal, and $\Sigma \in \mathbb{R}^{m\times n}$ is a nonsquare "diagonal" matrix

$$\Sigma = \begin{bmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_r & & \\ & & & & 0 & \\ & & & & & \ddots \end{bmatrix} \quad \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$$

1.2 Least square solution

1.3 Solution of least square problem

Consider an overdetermined system

$$Ax = b, \quad A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, m > n$$

Usually, $Ax = b$ does not have an exact solution. Thus, we try to find a solution $\hat{x}$ that minimizes $\|A\hat{x} - b\|_2$.

Theorem 1.2. *Let $A \in \mathbb{R}^{m\times n}, m > n$. Then there exist $Q \in \mathbb{R}^{m\times m}$ and $R \in \mathbb{R}^{m\times n}$, such that Q is orthogonal and $R = \begin{bmatrix} \hat{R} \\ 0 \end{bmatrix}$, where $\hat{R} \in \mathbb{R}^{n\times n}$ is upper triangular, and $A = QR$.*

Theorem 1.3. *Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m, m > n$, and suppose A has full rank. Then the least squares problem for the overdetermined system $Ax = b$ has a unique solution, which can be found by solving the nonsingular system $\hat{R}x = \hat{b}_1$, where $\begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \end{bmatrix} = Q^T b, \hat{R} \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{m\times m}$ are as in Theorem above.*

However, if $\operatorname{rank}(A) < n$, the solution is not unique; there are many x for which $\|Ax - b\|_2$ is minimized. To get the uniqueness, we consider the following additional problem: find the one for which $\|x\|_2$ is minimized. As we shall see, this problem always has a unique solution.

Theorem 1.4. *The LS solutions is $\hat{x} = V \cdot \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \end{bmatrix}$, where $\hat{y}_2 \in \mathbb{R}^{m-r}$ is free and $\hat{y}_1 = (\widehat{\Sigma})^{-1} \cdot \hat{c}$. Simply, the LS Solution set is:*

$$\left\{ \hat{x} = V \cdot \begin{bmatrix} (\widehat{\Sigma})^{-1} \cdot \hat{c} \\ \hat{y}_2 \end{bmatrix} \mid \hat{y}_2 \textit{ is free} \right\};$$

where $U^T b = c = \begin{bmatrix} \hat{c} \\ d \end{bmatrix}$

Thus,

- If $rank(A) = n$, the LS solution is unique and $\hat{x} = V\hat{y}_1$;
- If $rank(A) = r < n$, the solution with the minimum 2-norm is unique and can be computed by $\hat{x} = V \cdot \begin{bmatrix} \hat{y}_1 \\ 0 \end{bmatrix}$.

1.4 Pseudoinverse

Definition 1.5 (Pseudoinverse). *For $A \in \mathbb{R}^{m\times n}$, the Pseudoinverse of A is the matrix $A^\dagger \in \mathbb{R}^{n\times m}$ satisfying:*

1 $A \cdot A^\dagger \cdot A = A$

2 $\left(A \cdot A^\dagger\right)^T = A \cdot A^\dagger$

3 $A^\dagger \cdot A \cdot A^\dagger = A^\dagger$

4 $\left(A^\dagger \cdot A\right)^T = A^\dagger \cdot A$

Theorem 1.6. *For $A \in \mathbb{R}^{m\times n}$, if $A = U \cdot \Sigma \cdot V^T$ is the SVD, then the Pseudoinverse is given as*

$$A^\dagger = V \cdot \begin{bmatrix} (\widehat{\Sigma})^{-1} & 0 \\ 0 & 0 \end{bmatrix} \cdot U^T$$

We could also have the condensed version of the Pseudoinverse of A.

Theorem 1.7. *Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, and let $x \in \mathbb{R}^n$ be the minimum-norm solution of*

$$\|Ax - b\|_2$$

Then $x = A^{\dagger} b$.

Remark 1.8. *Just as the solution of a square nonsingular system $Ax = b$ can be expressed in terms of A^{-1} as $x = A^{-1}b$, the minimum-norm solution to a least squares problem with coefficient matrix $A \in \mathbb{R}^{m \times n}$ can be expressed in terms of the Pseudoinverse $A^{\dagger}$ as $x = A^{\dagger} b$.*

Exercise 1.9. *Show that if $A \in \mathbb{R}^{m \times n}, m \geq n$, and $\operatorname{rank}(A) = n$, then $A^{\dagger} = \left(A^T A\right)^{-1} A^T$*

Exercise 1.10. *For*

$$A = \begin{bmatrix} 2 & -4 \\ 3 & -6 \\ 4 & -8 \end{bmatrix} \; ; \; b = \begin{bmatrix} 4 \\ 5 \\ 4 \end{bmatrix}$$

1 *Find the (condensed) SVD of A.*

2 *Find the Pseudoinverse of A.*

3 *Find all solutions to the least-squares problem.*

4 *Find the minimum 2-norm solution of the least-squares.*