Discussion review 2

Math 181B

1 Review:

1.1 Wilk's theorem

Theorem 1.1 (Wilk's theorem)**.** *Let* $\Omega = \{\theta : \theta \in H_0 \cup H_1\}$ *and* $\Lambda = \frac{\max_{\theta \in H_0} L(\theta)}{\max_{\theta \in \Omega} L(\theta)}$. *No matter what distribution the sample data come from, under the null,* $-2 \log \Lambda$ *has asymptotic distribution* χ_k^2*, where* k *is the difference in dimensionality between* Ω *and* $\{\theta \in H_0\}$.

Corollary 1.1. *If* H_0 *is a single point which is an interior point of* H_1*,* $-2\log\Lambda$ *has asymptotic distribution* χ_k^2*, where* k *is the dimensionality of the space implied by* H_1.

1.2 F-distribution

- Let $U \sim \chi_n^2, V \sim \chi_m^2$ independent. Then $F = \frac{V/m}{U/n}$ has a **F**-distribution with m and n degrees of freedom. We write $F \sim F_{m,n}$.
- If $T \sim t_{n'}$, then $T^2 \sim F_{1,n'}$. The reason for this claim is simple: recall that $T = \frac{Z}{\sqrt{X/n}} \sim t_n$; where $X \sim \chi_n^2$. Then, $T^2 = \frac{Z^2}{X/n} = \frac{\chi_1^2/1}{\chi_n^2/n} \sim F_{1,n}$.

1.3 Confidence interval and HT for one-sample variance

Recall that $\frac{(n-1)S^2}{\sigma^2}$ has a chi-square distribution with $n-1$ degrees of freedom if our independent sample comes from a normal distribution, then, we have

$$P\left[\chi^2_{\alpha/2,n-1} \le \frac{(n-1)S^2}{\sigma^2} \le \chi^2_{1-\alpha/2,n-1}\right] = 1-\alpha.$$

In other words, a $100(1-\alpha)\%$ confidence interval for σ^2 based on some specific sample can be

$$\left[\frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}}, \frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}}\right];$$

where s^2 denote the sample variance calculated from a random sample of n observations drawn from a normal distribution with mean μ and variance σ^2.

We can also perform HT about one-sample variance. For example, to test $H_0 : \sigma^2 = \sigma_0^2$ versus $H_1 : \sigma^2 < \sigma_0^2$ at the α level of significance, we reject H_0 if $\chi^2 \le \chi^2_{\alpha,n-1}$; where $\chi^2 := \frac{(n-1)s^2}{\sigma_0^2}$ In addition, to test $H_0 : \sigma^2 = \sigma_0^2$ versus $H_1 : \sigma^2 \ne \sigma_0^2$ at the α level of significance, we reject H_0 if χ^2 is either (1) $\le \chi^2_{\alpha/2,n-1}$ or (2) $\ge \chi^2_{1-\alpha/2,n-1}$.

1.4 Confidence interval and HT for two-sample variance

Let $X_1, \ldots, X_n \sim N\left(\mu_X, \sigma_X^2\right)$ and $Y_1, \ldots, Y_m \sim N\left(\mu_Y, \sigma_Y^2\right)$ be two independent samples. Then $F = \frac{S_Y^2/\sigma_Y^2}{S_X^2/\sigma_X^2} \sim F_{m-1,n-1}$ (this is what you need to show by yourself later). Then, we have

$$P\left[F_{\frac{\alpha}{2}}(m-1,n-1) \leq \frac{\sigma_X^2}{\sigma_Y^2} \cdot \frac{S_Y^2}{S_X^2} \leq F_{1-\frac{\alpha}{2}}(m-1,n-1)\right] = 1-\alpha.$$

So a $100(1-\alpha)\%$ confidence interval for $\frac{\sigma_X^2}{\sigma_Y^2}$ can be:

$$\left[F_{\frac{\alpha}{2}}(m-1,n-1)\frac{s_X^2}{s_Y^2}\ ,\ F_{1-\frac{\alpha}{2}}(m-1,n-1)\frac{s_X^2}{s_Y^2}\right].$$

Also, we can do HT to compare the value of two variances. For example, to test $H_0 : \sigma_X^2 = \sigma_Y^2$ versus $H_1 : \sigma_X^2 < \sigma_Y^2$ at the α level of significance, we reject H_0 if $s_Y^2/s_X^2 \geq F_{1-\alpha,m-1,n-1}$.

1.5 Two-proportion z-interval/test

- A $100(1-\alpha)\%$ CI for $p_X - p_Y$ is:

$$\left[\frac{x}{n} - \frac{y}{m} + z_{\frac{\alpha}{2}}\sqrt{\frac{\frac{x}{n}\left(1-\frac{x}{n}\right)}{n} + \frac{\frac{y}{m}\left(1-\frac{y}{m}\right)}{m}}\ ,\ \frac{x}{n} - \frac{y}{m} - z_{\frac{\alpha}{2}}\sqrt{\frac{\frac{x}{n}\left(1-\frac{x}{n}\right)}{n} + \frac{\frac{y}{m}\left(1-\frac{y}{m}\right)}{m}}\right]$$

- Test statistics can be:

$$Z = \frac{\frac{X}{n} - \frac{Y}{m}}{\sqrt{\frac{p_e(1-p_e)}{n} + \frac{p_e(1-p_e)}{m}}} \sim N(0,1);$$

  where $p_e = \frac{x+y}{n+m}$ (the maximum likelihood estimator under null, i.e., assuming $p_X = p_Y$).

- P-value can be computed by $P(Z \leq z)$ or $P(Z \geq z)$ or $2P(Z \leq -|z|)$.