Discussion section 7

Math 181B

1 Review

1.1 F-test

Our goal is to test $H_0 : \mu_1 = \mu_2 = \ldots = \mu_k$ and H_1 : At least one μ_i is different. Also, we assume that Y_{ij} are independent and normally distributed with mean $\mu_j, j = 1, 2, \ldots, k$, and variance σ^2(constant for all j). Suppose, for each level, we have independent samples with size $n_1, \ldots, n_k$. Some useful statistics are listed below:

- Treatment sum of squares:

$$\text{SSTR} = \sum_{j=1}^{k}\sum_{i=1}^{n_j}\left(\bar{Y}_{.j} - \bar{Y}_{..}\right)^2 = \sum_{j=1}^{k} n_j\left(\bar{Y}_{.j} - \bar{Y}_{..}\right)^2 = \sum_{j=1}^{k} n_j\left(\bar{Y}_{.j} - \mu\right)^2 - n\left(\bar{Y}_{..} - \mu\right)^2 .$$

- Error sum of squares:

$$\text{SSE} = \sum_{j=1}^{k}\left(n_j - 1\right) S_j^2 = \sum_{j=1}^{k}\sum_{i=1}^{n_j}\left(Y_{ij} - \bar{Y}_{.j}\right)^2 .$$

- Total sum of squares:

$$\text{SSTOT} = \sum_{j=1}^{k}\sum_{i=1}^{n_j}\left(Y_{ij} - \bar{Y}_{..}\right)^2 .$$

Remark:

- $\text{SSTOT} = \text{SSTR} + \text{SSE}$.
- $E(\text{SSTR}) = (k-1)\sigma^2 + \sum_{j=1}^{k} n_j\left(\mu_j - \mu\right)^2$.
- Under $H_0 : \mu_1 = \mu_2 = \ldots = \mu_k$ is true, SSTR $/\sigma^2 \sim \chi^2_{k-1}$ (proof based on Moment generating functions).
- Whether or not $H_0 : \mu_1 = \mu_2 = \ldots = \mu_k$ is true, $\text{SSE}/\sigma^2 \sim \chi^2_{n-k}$ (proof based on Theorem 7.3.2 in the textbook).
- Whether or not $H_0 : \mu_1 = \mu_2 = \ldots = \mu_k$ is true, SSE and SSTR are independent.
- Under $H_0 : \mu_1 = \mu_2 = \ldots = \mu_k$ is true, $\frac{\text{SSTOT}}{\sigma^2} \sim \chi^2_{n-1}$.
- When σ^2 is unknown, under H_0, we can take $F = \frac{\text{SSTR}/(k-1)}{\text{SSE}/(n-k)}$ which has an F distribution with $k-1$ and $n-k$ degrees of freedom.

1.2 ANOVA Table

Source	df	SS	MS	F	p-value
Treatment	$k-1$	SSTR	MSTR	$\frac{MSTR}{MSE}$	$P\left(F_{k-1,n-k} \geq \text{ observed } F\right)$
Error	$n-k$	SSE	MSE		
Total	$n-1$	SSTOT			

where, $MSTR = \frac{SSTR}{k-1}; MSE = \frac{SSE}{n-k}$.

1.3 Relations to t-test

When we only have two groups of normal samples that share the same variance σ^2, then the F-test is equivalent to the two-sample t-test in the below way:

$$\alpha = P\left(T \leq t_{\alpha/2,n_1+n_2-2} \quad or \quad T \geq -t_{\alpha/2,n_1+n_2-2}\right) = P\left(F \geq F_{1-\alpha,1,n_1+n_2-2}\right)$$

1.4 Multiple Comparisons and Bonferroni correction

We should notice that the F-test checks if we have the confidence to reject that all means of different groups of samples are the same, which means there is at least one mean value that is different from others. We may do such a check by performing two-sample t-test one by one.

However, we need to be careful to control the type-I error for the whole test procedure. For example, if we do 10 two-sample t-tests and apply confidence level $\alpha = 0.05$ for all tests. The probability of making at least one type-I error can be computed as $P(\text{ at least one Type I error }) = 1 - P(\text{ no Type I errors }) = 1 - (0.95)^{10} = 0.40$, which is pretty large.

Then, we can apply the Bonferroni correction, i.e., we set the confidence level $\alpha = 0.05/m$, m is the number of tests we wanted to perform. Denote A_i the event we make type-I error for the i-th test, i.e., $P(A_i) = 0.05/m$, then we have:

$$P(\text{ at least one Type I error }) = P(A_1 or A_2 \cdots or A_m) = P(\bigcup_{i=1}^{m} A_i) \leq \sum_{i=1}^{m} P(A_i) = \alpha.$$